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Journal of Computational and Applied Mathematics 179 (2005) 195–213
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Orthogonal Laurent polynomials and two-point Padé approximants associated with Dawson's integral ☆

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Received 27 November 2003

Dedicated to Olav Njåstad on the occasion of his 70th birthday

Abstract

Starting from the coefficients of the power expansions for the Dawson's integral both around the origin and infinity, a linear functional acting on the space of Laurent polynomials is defined. In this paper, properties of this functional are studied in connection with certain sequences of orthogonal Laurent polynomials and two-point Padé approximants.

Keywords: Dawson's integral; Strong moment functional; Orthogonal Laurent polynomial; Two-point Padé approximant

1. Introduction

When trying to develop a general theory concerning orthogonal rational functions with prescribed poles, as in [2], and that enables us to extend most of the known properties on orthogonal polynomials, a simple fact should be recalled: *a polynomial can be viewed as a rational function with all of its poles located at infinity*. Thus, it seems quite natural the next step to be given leads to rational functions with prescribed poles at the origin and infinity (take into account this situation is not considered in [2]). In this form, the so-called Laurent polynomials (L-polynomials) appear, giving rise to a theory that parallels

☆ This work was supported by the Scientific Research Projects of the Ministerio de Ciencia y Tecnología and Comunidad Autónoma de Canarias under contracts BFM2001-3411 and PI 2002/136, respectively.
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0377-0427/$ - see front matter
doi:10.1016/j.cam.2004.09.041

rather closely that one corresponding to usual orthogonal polynomials, and whose starting point can be considered the excellent paper by Jones et al. [13]. As indicated in [5], after that paper many contributions have been given on this topic, playing a fundamental role the works by O. Njåstad and collaborators, very specially in the solvability of the so-called *strong Hamburger moment problem*, [12,20,22], as a natural continuation of the problem solved in [13] (as for a general theory on orthogonal L-polynomials see also [21]).

On the other hand, it should be also pointed out that in a similar way as orthogonal polynomials are closely related to one-point Padé approximants (at infinity), see e.g. [1], orthogonal Laurent polynomials are connected with two-point Padé approximants at the origin and infinity [11], both topics being the main ingredients of this paper. Furthermore, when speaking about orthogonality for Laurent polynomials, within an algebraic general framework, a double sequence $\{\mu_k\}_{k=-\infty}^{\infty}$ of real or complex numbers which represents the so-called *strong moments* will be needed. In this paper, such sequence will be associated with the well known Dawson's integral

$$D(x) = \mathrm{e}^{-x^2} \int_0^x \mathrm{e}^{t^2}\, \mathrm{d}t.$$

The function D was first tabulated by Dawson [6] and since then, extensive tabulations have appeared (see e.g. [16] and references therein) because of its applications in several physical problems as heat conduction, spectroscopy and electrical oscillations.

The paper has been organized as follows. In Section 2, along with some previous results which will be needed later on, a general sequence of orthogonal Laurent polynomials is introduced motivating the concept of a quasi-definite strong moment functional. The connection between two-point Padé approximants and orthogonal Laurent polynomials is again revisited. In Section 3, the coefficients of the McLaurin series and an asymptotic expansion as $z \to \infty$ for the Dawson's integral enable us to define a strong moment functional whose properties are analyzed in Section 4. Special emphasis is done on the location of the zeros of certain sequence of orthogonal Laurent polynomials. Finally, some conclusions and numerical experiments are given in Section 5.

2. Preliminary results

Given two integers p and q such that $p \leqslant q$, we denote by $\Delta_{p,q}$ the linear space of Laurent polynomials,

$$\Delta_{p,q} = \operatorname{span}\, \langle x^j \,:\, p \leqslant j \leqslant q \rangle,$$

and by Δ the space of all Laurent polynomials. Also for a nonnegative integer n, Π_n will denote the space of polynomials of degree n at most and Π the space of all polynomials. Thus, in order to handle a certain nested sequence of subspaces of Laurent polynomials, similar to the sequence $\{\Pi_k\}_{k \geqslant 0}$ in Π, we will start from a nondecreasing sequence $\{p(n)\}_{n \geqslant 0}$ of nonnegative integers such that,

(i) $0 \leqslant p(n) \leqslant n, \ n = 0, 1, \ldots,$
(ii) $s(n) = p(n) - p(n-1) \in \{0, 1\}, \ n = 1, 2, \ldots,$

and define $\mathscr{R}_n = \Delta_{-p(n),q(n)} = \operatorname{span}\, \langle x^j \,:\, -p(n) \leqslant j \leqslant q(n) \rangle$ with $q(n) = n - p(n)$. Now, we have $\dim(\mathscr{R}_n) = n + 1$ and $\mathscr{R}_n \subset \mathscr{R}_{n+1}$, $n = 0, 1, 2, \ldots$. Furthermore, by setting $\mathscr{R} = \bigcup_0^{\infty} \mathscr{R}_n$, then $\mathscr{R} = \Delta$ if and only if $\lim_{n\to\infty} p(n) = \lim_{n\to\infty} q(n) = \infty$. In any case, it will be said that $\{p(n)\}_{n\geqslant 0}$ has generated or induced an *ordering* in $\mathscr{R}$.

Let $\{\mu_k\}_{k=-\infty}^{\infty}$ be a given double sequence of real or complex numbers giving rise to a linear functional $\mathscr{L}$ on Δ, as follows,

$$\mathscr{L} : \Delta \longrightarrow \mathbb{C}, \quad \mathscr{L}(x^k) = \mu_k, \quad k = 0, \pm 1, \pm 2, \dots .$$

Now, a sequence of Laurent polynomials $\{R_n\}_{n\geqslant 0}$ will be said an *orthogonal Laurent polynomial sequence* (OLPS in short) with respect to the double sequence $\{\mu_k\}_{k=-\infty}^{\infty}$ (or the functional $\mathscr{L}$, which is called a *strong moment functional* (SMF)) and the ordering induced by $\{p(n)\}_{n\geqslant 0}$, if and only if the following holds,

(i) For $n = 1, 2, \dots$, $R_n \in \mathscr{R}_n \backslash \mathscr{R}_{n-1}$ and $R_0 \in \mathscr{R}_0$, i.e. if we write $R_n(x) = \sum_{j=-p(n)}^{q(n)} r_{n,j}\, x^j$ and set

$$\gamma_n = \begin{cases} r_{n,-p(n)} & \text{if } s(n) = 1, \\ r_{n,q(n)} & \text{if } s(n) = 0, \end{cases}$$

then $\gamma_n \neq 0$ for each n. γ_n is called the leading coefficient of R_n. When $\gamma_n = 1$, R_n will be said *monic*.

(ii) $\mathscr{L}(R_n R_m) = \Omega_n \delta_{n,m}$, with $\Omega_n \neq 0$ and $\delta_{n,m} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$

Now, for a given SMF $\mathscr{L}$ and fixed a certain ordering induced by $\{p(n)\}_{n\geqslant 0}$, a question immediately arises: under which conditions, does there exist an OLPS?

First, observe that condition (ii) is equivalent to $(n = 1, 2, \dots)$

$$\mathscr{L}(r\, R_n) = 0, \ \forall r \in \mathscr{R}_{n-1} \quad \text{and} \quad \mathscr{L}(r\, R_n) \neq 0, \ \forall r \in \mathscr{R}_n \backslash \mathscr{R}_{n-1},$$

i.e.

$$\mathscr{L}(x^k R_n) = 0, \quad -p(n-1) \leqslant k \leqslant q(n-1) \tag{1}$$

and

$$\mathscr{L}(x^{-p(n)} R_n) = K_n \neq 0 \quad \text{if } s(n) = 1 \tag{2}$$

or

$$\mathscr{L}(x^{q(n)} R_n) = K_n \neq 0 \quad \text{if } s(n) = 0. \tag{3}$$

The existence theorem will be given in terms of Hankel determinants associated with the double sequence $\{\mu_k\}_{k=-\infty}^{\infty}$, i.e.

$$H_0^{(n)} = 1, \quad H_k^{(n)} = \begin{vmatrix} \mu_n & \mu_{n+1} & \cdots & \mu_{n+k-1} \\ \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n+k-1} & \mu_{n+k} & \cdots & \mu_{n+2k-2} \end{vmatrix}, \quad n \in \mathbb{Z}, \ k \in \mathbb{N}.$$

Theorem 1 (*existence theorem*). *Let $\mathscr{L}$ be a SMF associated with the moment double sequence $\{\mu_k\}_{k=-\infty}^{\infty}$ and $\{p(n)\}_{n\geqslant 0}$ a nondecreasing sequence of nonnegative integers inducing an ordering in $\mathscr{R}$. Then,* *there exists an OLPS for $\mathscr{L}$ if and only if,*

$$H_n^{(-2p(n-1))} \neq 0, \quad n = 1, 2, \ldots . \tag{4}$$

Proof. Take $R_n \in \mathscr{R}_n$, so that we can write $R_n(x) = \sum_{j=-p(n)}^{q(n)} r_{n,j} x^j$. By (1), if denote $\alpha_{n,j} = r_{n,-p(n)+j}$, we have,

$$\sum_{j=0}^{n} \alpha_{n,j} \mu_{-p(n)+j+k} = 0, \quad -p(n-1) \leqslant k \leqslant q(n-1). \tag{5}$$

Thus, $\{\alpha_{n,j}\}_{j=0}^{n}$, satisfies a homogeneous linear system with n equations and $n+1$ unknowns. Hence, a nontrivial solution there always exists. Now from (2) and (3), we have

$$\mathscr{L}\left(\left(\frac{s(n)}{x^{p(n)}} + [1-s(n)]x^{q(n)}\right) R_n\right) = K_n,$$

or equivalently,

$$\sum_{j=0}^{n} \alpha_{n,j} \mu_{-2s(n)p(n)+[1-s(n)](-2p(n)+n)+j} = K_n. \tag{6}$$

So, we see that (5) and (6) represents a linear system with $n+1$ equations and $n+1$ unknowns. Hence, this linear system admits a unique solution if and only if the determinant of the coefficient matrix

$$A_n = s(n) \begin{pmatrix} \mu_{-2p(n)+1} & \mu_{-2p(n)+2} & \cdots & \mu_{-2p(n)+n+1} \\ \mu_{-2p(n)+2} & \mu_{-2p(n)+3} & \cdots & \mu_{-2p(n)+n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{-2p(n)+n} & \mu_{-2p(n)+n+1} & \cdots & \mu_{-2p(n)+2n} \\ \mu_{-2p(n)} & \mu_{-2p(n)+1} & \cdots & \mu_{-2p(n)+n} \end{pmatrix} + [1-s(n)] \begin{pmatrix} \mu_{-2p(n)} & \mu_{-2p(n)+1} & \cdots & \mu_{-2p(n)+n} \\ \mu_{-2p(n)+1} & \mu_{-2p(n)+2} & \cdots & \mu_{-2p(n)+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{-2p(n)+n} & \mu_{-2p(n)+n+1} & \cdots & \mu_{-2p(n)+2n} \end{pmatrix},$$

is nonzero. It can be easily seen that $\det(A_n) = (s(n)(-1)^n + [1-s(n)]) H_{n+1}^{(-2p(n))}$.

Now, it remains to see that $\alpha_{n,[1-s(n)]n} \neq 0$. Indeed, using Cramer's rule to solve (5) and (6) for $\alpha_{n,[1-s(n)]n}$ will yield

$$\alpha_{n,[1-s(n)]n} = K_n \frac{H_n^{(-2p(n-1))}}{H_{n+1}^{(-2p(n))}},$$

and the proof follows. □

Remark 2. If an OLPS exists, it is uniquely determined up to an arbitrary nonzero constant. Thus, the monic OLPS for $\mathscr{L}$ is unique.

Remark 3. Once more again, it should be recalled that we are starting from a fixed ordering in Δ, generated by a sequence $\{p(n)\}_{n\geqslant 0}$. In this respect, some of the most usual selections of $p(n)$ appearing in the literature are,

- $p(n)=0$, $q(n)=n$, $\mathscr{R}_n=\Pi_n$. Now $s(n)=0$. Therefore, (4) becomes $H_n^{(0)}\neq 0$, $n=1,2,\ldots$. Thus, the well known property of existence of orthogonal polynomials in terms of Hankel determinants is recovered (see e.g. [4]).
- $p(n)=\mathrm{E}[(n+1)/2]$, which gives rise to the following sequence of nested subspaces

  $$\Delta_{0,0},\ \Delta_{-1,0},\ \Delta_{-1,1},\ \Delta_{-2,1},\ \ldots$$

  (recall that most of the contributions on orthogonal Laurent polynomials refer to this ordering. See e.g. [5] and references therein). Now when n is even, then $2p(n)=n$ and $2p(n)=n+1$, otherwise. Hence, (4) becomes,

  $$H_{2m}^{(-2m)}H_{2m+1}^{(-2m)}\neq 0,\quad m=0,1,2,\ldots$$

  (compare with Theorem 2.2 in [5]).
- $p(n)=\mathrm{E}[n/2]$. This selection has been also handled in the literature on orthogonal Laurent polynomials (see e.g. [3]), giving now rise to the following sequence of linear subspaces

  $$\Delta_{0,0},\ \Delta_{0,1},\ \Delta_{-1,1},\ \Delta_{-1,2},\ \Delta_{-2,2},\ \ldots .$$

  Now, when n is even then $2p(n)=n$ and $2p(n)=n-1$, otherwise. Thus, the determinantal condition (4) now becomes,

  $$H_{2m}^{(-2m+2)}H_{2m+1}^{(-2m)}\neq 0,\quad m=0,1,2,\ldots .$$

Now, some technical definitions to be handled later on will be introduced. Thus, fixed an ordering induced by $\{p(n)\}_{n\geqslant 0}$, a strong moment functional $\mathscr{L}$ for which an OLPS exists will be called *quasi-definite* with respect to such ordering. Assume that $\{R_n\}_{n\geqslant 0}$ is an OLPS with respect to a quasi-definite SMF $\mathscr{L}$ and the ordering induced by $\{p(n)\}_{n\geqslant 0}$, and set $R_n(x)=\sum_{j=-p(n)}^{q(n)} r_{n,j}x^j$, then, R_n is said to be *regular* if $r_{n,-p(n)}r_{n,q(n)}\neq 0$, and $\{R_n\}_{n\geqslant 0}$ is *regular* if R_n is regular for each $n=1,2,\ldots$. Observe that if $R_n(x)=x^{-p(n)}P_n(x)$, $P_n\in\Pi_n$, then R_n is regular if and only if, P_n has exact degree n and $P_n(0)\neq 0$. Furthermore, there holds,

Theorem 4. *Let $\mathscr{L}$ be a quasi-definite SMF with respect to the moment double sequence $\{\mu_k\}_{k=-\infty}^{\infty}$ and $\{p(n)\}_{n\geqslant 0}$ a nondecreasing sequence of nonnegative integers inducing an ordering in $\mathscr{R}$. Then, R_n is regular if only if,*

$$H_n^{(-2p(n)+1)}\neq 0.$$

Proof. From the systems (5) and (6) it can be easily seen that

$$\alpha_{n,s(n)n}=(-1)^n K_n\,\frac{H_n^{(-2p(n)+1)}}{H_{n+1}^{(-2p(n))}},$$

and proof trivially follows. $\square$

Next, and for the sake of completeness, in the rest of this section we will concentrate ourselves on the connection between OLPs and the so-called *two-point Padé approximants*. Indeed, from a given strong moment sequence $\{\mu_k\}_{k=-\infty}^{\infty}$ (or from the SMF $\mathscr{L}$), we can associate the formal expansions (see [21] for details)

$$\begin{aligned} L_0(z) &= -\sum_{j=0}^{\infty} \mu_{-(j+1)} z^j, \quad (z \to 0), \\ L_\infty(z) &= \sum_{j=1}^{\infty} \mu_{j-1} z^{-j}, \quad (z \to \infty), \end{aligned} \tag{7}$$

so that, for nonnegative integers n and k such that $0 \leqslant k \leqslant 2n$, there exist polynomials $P_{k,n}$ ($P_{k,n} \not\equiv 0$) and $Q_{k,n}$ of degree at most n and $n-1$, respectively, such that

$$\begin{aligned} P_{k,n}(z)L_0(z) - Q_{k,n}(z) &= d^*_{k,n} z^k + d^*_{k+1,n} z^{k+1} + \ldots = \mathcal{O}(z^k), \\ P_{k,n}(z)L_\infty(z) - Q_{k,n}(z) &= \frac{d_{n+1-k,n}}{z^{n+1-k}} + \frac{d_{n+2-k,n}}{z^{n+2-k}} + \ldots = \mathcal{O}\left(\frac{1}{z^{n+1-k}}\right). \end{aligned} \tag{8}$$

Furthermore, any solution $(P_{k,n}, Q_{k,n})$ defines a unique rational function $Q_{k,n}/P_{k,n}$ called the $[k/n]$-two-point Padé approximant (2PA in short) to the pair (L_0, L_∞), in the *weak sense*.

Clearly, when $P_{k,n}$ has exact degree n and $P_{k,n}(0) \neq 0$, from (8), one has,

$$\begin{aligned} L_0(z) - \frac{Q_{k,n}(z)}{P_{k,n}(z)} &= \mathcal{O}(z^k), \quad (z \to 0), \\ L_\infty(z) - \frac{Q_{k,n}(z)}{P_{k,n}(z)} &= \mathcal{O}\left(\frac{1}{z^{2n+1-k}}\right), \quad (z \to \infty). \end{aligned} \tag{9}$$

In this case, $Q_{k,n}/P_{k,n}$ is said the $[k/n]$-2PA in the *strong sense*. In both cases, it will be denoted by $[k/n]_{(L_0,L_\infty)}$.

Now, concerning the construction of the polynomials $P_{k,n}$ and $Q_{k,n}$, it can be easily seen from (8) that ([9])

$$\mathscr{L}(x^{j-k} P_{k,n}(x)) = 0, \quad 0 \leqslant j \leqslant n-1. \tag{10}$$

Suppose now that $\mathscr{L}$ is quasi-definite with respect to the ordering induced by a sequence $\{p(n)\}_{n \geqslant 0}$. Set $s(n) = p(n) - p(n-1) \in \{0, 1\}$ and $k = k(n) = p(n) + p(n-1)$, and consider the 2PA $[k(n)/n]_{(L_0,L_\infty)}(z) = Q_n(z)/P_n(z)$, then from (10), and setting $R_n(x) = x^{-p(n)} P_n(x) \in \mathscr{R}_n$, one has

$$\mathscr{L}(x^{j-p(n-1)} R_n(x)) = 0, \quad 0 \leqslant j \leqslant n-1,$$

yielding

$$\mathscr{L}(x^r R_n(x)) = 0, \quad -p(n-1) \leqslant r \leqslant q(n-1). \tag{11}$$

On the other hand, define

$$\widetilde{R}_n(z) = \mathscr{L}\left(\frac{R_n(z) - R_n(x)}{z - x}\right) \tag{12}$$

(the functional $\mathscr{L}$ acting on the variable x and z being a parameter). Then, it can be checked that $\widetilde{R}_n(z) = z^{-p(n)}\widetilde{Q}_n(z)$, where $\widetilde{Q}_n$ is a polynomial of degree at most $n-1$ satisfying the conditions (8) with $k=k(n)$. Thus, we have come to the following,

Theorem 5. *Assume that the SMF $\mathscr{L}$ is quasi-definite with respect to the sequence $\{p(n)\}_{n\geqslant 0}$ of non-negative integers. Let $\{R_n\}_{n\geqslant 0}$ be an OLPS for $\mathscr{L}$, then*

$$[k(n)/n]_{(L_0,L_\infty)}(z) = \frac{\widetilde{R}_n(z)}{R_n(z)},$$

where $k(n) = p(n) + p(n-1)$, $n = 1, 2, \ldots$ and $\widetilde{R}_n$ is given by (12).

There also holds,

Corollary 6. *Under the same assumptions as in Theorem* 5, *if R_n is regular, then the $[k(n)/n]$-2PA also exists in the strong sense.*

Conversely, let (L_0, L_∞) be a pair of formal expansions like (7), i.e.

$$L_0(z) = \sum_{j=0}^{\infty} c_j z^j, \quad L_\infty(z) = \sum_{j=1}^{\infty} c_j^* z^{-j}, \quad c_j,\, c_j^* \in \mathbb{C}. \tag{13}$$

We can now associate a double sequence $\{\mu_k\}_{k=-\infty}^{\infty}$ as follows:

$$\begin{aligned} &\mu_{-k} = -c_{k-1}, \quad k = 1, 2, \ldots, \\ &\mu_k = c_{k+1}^*, \quad k = 0, 1, \ldots, \end{aligned} \tag{14}$$

and let $\mathscr{L}$ be the corresponding SMF, i.e. $\mathscr{L}(x^k) = \mu_k$, $k \in \mathbb{Z}$. Let $\{p(n)\}_{n\geqslant 0}$ be a given sequence of nonnegative integers such that $s(n) = p(n) - p(n-1) \in \{0, 1\}$, and set $k(n) = p(n) + p(n-1)$, then it can be easily proved the following.

Theorem 7. *Let $[k(n)/n]_{(L_0,L_\infty)} = Q_n/P_n$. Set $R_n(z) = z^{-p(n)}P_n(z)$ and $\widetilde{R}_n(z) = z^{-p(n)}Q_n(z)$, then there holds,*

(i) $\widetilde{R}_n(z) = \mathscr{L}\left(\frac{R_n(z)-R_n(x)}{z-x}\right)$.
(ii) $\mathscr{L}(x^j R_n) = 0, -p(n-1) \leqslant j \leqslant q(n-1)$.

To conclude this section it should be indicated that a class of SMF that is of special interest is the one consisting of functionals $\mathscr{L}$ which can be represented by

$$\mu_n = \mathscr{L}(x^n) = \int_a^b x^n \, \mathrm{d}\psi(x), \quad n \in \mathbb{Z}, \quad -\infty \leqslant a < b \leqslant \infty, \tag{15}$$

where ψ is a *strong distribution function*, i.e. a real valued, nondecreasing function with infinitely many points of increase on (a, b) such that the integrals (15) exist. When $(a, b) = (0, \infty)$, then strong Stieltjes distributions arise (see [13]). In a recent paper [7], the authors carry on a systematic study of general sequences of OLP and related topics as quadratures and 2PA.

Clearly, for a SMF given by (15), and $R \in \Delta$ such that $R(x) \geqslant 0$, $x \in \mathbb{R}\backslash\{0\}$, there holds $\mathscr{L}(R) > 0$. This gives rise to the following definition: a SMF $\mathscr{L}$ is said to be *positive definite* if and only if $\mathscr{L}(R) > 0$, for any $R \in \Delta\,(R \not\equiv 0)$ such that $R(x) \geqslant 0$, $x \in \mathbb{R}\backslash\{0\}$. Observe that, unlike the given definition of a quasi-definite SMF $\mathscr{L}$, the definition of positive definiteness is independent of the ordering fixed on Δ. Furthermore, if $\mathscr{L}$ is positive definite, then a real inner product can be defined in Δ, as usual i.e. $\langle R, S\rangle = \mathscr{L}(R \cdot S)$. Thus, making use of the Gram–Schmidt process, one can prove

Proposition 8. *If the SMF $\mathscr{L}$ is positive definite, then $\mathscr{L}$ is quasi-definite with respect to any ordering.*

Certainly, and as for as we know, very few examples of SMF to be quasi-definite and not positive definite have been considered. This will be done in the next sections, where a SMF generated from Dawson's integral will be considered and its most relevant properties analyzed with respect to the ordering induced by the sequence $p(n) = \mathrm{E}[(n+1)/2]$.

3. A SMF associated with Dawson's integral

The function

$$F(p, x) = \mathrm{e}^{-x^p} \int_0^x \mathrm{e}^{t^p}\,\mathrm{d}t, \quad x \geqslant 0,\ \ p > 0,$$

is a generalization of the so-called Dawson's integral ([8]) obtained for $p = 2$. The function $F(3, x)$ occurs in viscous fluid mechanic, meanwhile, and as already mentioned, Dawson's integral is of a great interest in other several physical problems.

In this section we shall be concerned with the function $D(x) = F(2, x)$ in order to deduce a SMF associated with it.

Thus, we can first obtain the McLaurin series for D, i.e. (see e.g. [16] for details)

$$D(z) = \sum_{k=0}^{\infty} (-1)^k \frac{k!\,2^{2k}}{(2k+1)!} z^{2k+1}, \tag{16}$$

that converges for all finite z, that is, D is an entire function.

On the other hand, D admits the asymptotic expansion ([15]),

$$\begin{aligned} D(z) &\sim \left(\frac{1}{2z} + \frac{1}{4z^3} + \frac{3}{8z^5} + \dots\right) - \frac{\mathrm{i}\sqrt{\pi}}{2}\,\varepsilon \exp(-z^2) \\ &= \frac{1}{2z}\,{}_2F_0(1, 2^{-1}, z^{-2}) - \frac{\mathrm{i}\sqrt{\pi}}{2}\,\varepsilon \exp(-z^2) \end{aligned} \tag{17}$$

for $|z|$ large and $-(3+2\varepsilon)\,\frac{\pi}{4} < \arg(z) < (3-2\varepsilon)\,\frac{\pi}{4}$, $\varepsilon = \pm 1$.

For our purposes, instead of dealing with the function D we shall handle the function G defined by

$$D(z) = z\,G(2z^2). \tag{18}$$

Thus, from (16), it can be deduced that the power series expansion

$$G_0(z) = 1 - \frac{z}{3} + \frac{z^2}{15} - \frac{z^3}{105} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^k}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k+1)} z^k, \tag{19}$$

represents the McLaurin series of the entire function G.

On the other hand, by setting

$$G_\infty(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{3}{z^3} + \frac{15}{z^4} + \ldots = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)}{z^{k+1}}, \tag{20}$$

it can be also checked from (17) that $zG_\infty(2z^2)$ is an asymptotic expansion for D valid in $|\arg(z)| < \frac{\pi}{4}$.

In order to introduce the SMF to be analyzed, let us consider the power expansions given by (19) and (20), so that, according to (13) and (14), one can define the following SMF

$$\mathscr{L} : \varDelta \longrightarrow \mathbb{R}, \quad \mathscr{L}(x^k) = \mu_k, \quad k \in \mathbb{Z},$$

such that

$$\begin{aligned} &\mu_{-k} = -c_{k-1} = \frac{(-1)^k}{1 \cdot 3 \cdot 5 \cdots (2k-1)}, \quad k = 1, 2, \ldots, \\ &\mu_0 = 1, \\ &\mu_k = c^*_{k+1} = 1 \cdot 3 \cdot 5 \cdots (2k-1), \quad k = 1, 2, \ldots . \end{aligned} \tag{21}$$

In the next section we will try to investigate the properties of this SMF with respect to the ordering induced by the sequence $p(n) = \mathrm{E}[(n+1)/2]$. For it, we will take advantage of certain 2PA to the pair (G_0, G_∞) through the theory of continued fractions instead of directly considering the Hankel determinants associated with the double sequence (21). In this respect, an answer to the following questions will be given: is the SMF $\mathscr{L}$ quasi-definite?, If so, is it positive definite?. Furthermore, a result concerning location of the zeros of the corresponding OLPS will be also provided with.

4. Orthogonal Laurent polynomials and two-point Padé approximants

Let $\mathscr{L}$ be the SMF given by (21) and consider the ordering in $\varDelta = \mathscr{R}$ generated by the sequence $p(n) = \mathrm{E}[(n+1)/2]$.

From the power expansions (19) and (20) for G given by (18), and proceeding as in [21] (see also [17,19]), one can easily check that G admits the continued fraction

$$G(z) = \frac{1}{1+z} \, \underset{-}{} \, \frac{2z}{3+z} \, \underset{-}{} \, \frac{4z}{5+z} \, \underset{-}{} \, \ldots \, \underset{-}{} \, \frac{2(n-1)z}{(2n-1)+z} \, \underset{-}{} \, \ldots .$$

This continued fraction is called an M-fraction and was introduced by McCabe and Murphy (see e.g. [17–19]).

Certainly, G can be also expressed like

$$G(z) = \frac{\alpha_1}{z+\beta_1} \, \underset{+}{} \, \frac{\alpha_2 z}{z+\beta_2} \, \underset{+}{} \, \frac{\alpha_3 z}{z+\beta_3} \, \underset{+}{} \, \ldots \, \underset{+}{} \, \frac{\alpha_n z}{z+\beta_n} \, \underset{+}{} \, \ldots, \tag{22}$$

where

$$\alpha_1 = 1, \quad \alpha_n = -2(n-1), \; n \geqslant 2,$$
$$\beta_n = 2n - 1, \; n \geqslant 1. \tag{23}$$

Let G_n denote the nth convergent of (22). Then, one has $G_n = A_n/B_n$, where A_n and B_n are polynomials of degree at most $n-1$ and n, respectively and there holds (from the elementary theory of continued fractions, see e.g. [14]),

$$A_{n+1}(z) = (z + \beta_{n+1})A_n(z) + \alpha_{n+1} z A_{n-1}(z),$$
$$B_{n+1}(z) = (z + \beta_{n+1})B_n(z) + \alpha_{n+1} z B_{n-1}(z), \tag{24}$$

with initial conditions $A_{-1}(z) = z^{-1}$, $A_0 \equiv 0$, $B_{-1} \equiv 0$ and $B_0 \equiv 1$. From (24) one clearly sees that for $n \geqslant 1$, B_n and A_n are polynomials of exact degree n and $n-1$, respectively. Furthermore, $B_n(0) \neq 0$, or more precisely, for $n \geqslant 1$, one has

$$B_n(0) = \beta_1 \ldots \beta_n > 0. \tag{25}$$

On the other hand, from the correspondence property of M-fractions associated with the pair (G_0, G_∞) (see [18]), there holds

$$G_0(z) - G_n(z) = \mathcal{O}(z^n), \quad (z \to 0),$$
$$G_\infty(z) - G_n(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \quad (z \to \infty),$$

and consequently, taking into account (9), by virtue of uniqueness, G_n coincides with the $[n/n]$-2PA to (G_0, G_∞), i.e. one can write

$$[n/n]_{(G_0, G_\infty)} = G_n = \frac{A_n}{B_n}.$$

Thus, in the sequel, we can refer to the $[n/n]$-2PA to (G_0, G_∞) as the quotient A_n/B_n with A_n and B_n satisfying (24). Furthermore, there holds,

Proposition 9. *For each $n \geqslant 1$, the $[n/n]$-2PA to the pair (G_0, G_∞) given by* (19) *and* (20) *exists in the strong sense.*

Remark 10. The L-polynomial $R_n(x) = B_n(x)/x^{E[(n+1)/2]}$ is regular and, from Theorem 7, satisfies $\mathscr{L}(r \cdot R_n) = 0$, $r \in \mathscr{R}_{n-1}$, $n \geqslant 1$.

Now, in order to study the SMF $\mathscr{L}$ associated with the Dawson's integral, the following Favard's theorem (see [5]) will be required. A Favard theorem for Laurent polynomials is already in [10], but with a different normalization.

Theorem 11. *Let $\{a_n\}_{n \geqslant 1}$ and $\{b_n\}_{n \geqslant 1}$ be sequences of nonzero complex numbers related by $b_n = a_{n+1}$, for $n \geqslant 1$ ($a_1 = 1$), and let $\{\lambda_n\}_{n \geqslant 1}$ be a sequence of complex numbers. Define $\{\tilde{R}_n\}_{n \geqslant 0}$ by setting $\tilde{R}_{-1} \equiv 0$, $\tilde{R}_0 \equiv 1$ and*

$$\tilde{R}_{2n+1}(z) = \left(\frac{z^{-1}}{a_{2n+1}} + b_{2n+1}\right)\tilde{R}_{2n}(z) + \lambda_{2n+1}\tilde{R}_{2n-1}(z),$$

$$\tilde{R}_{2n+2}(z) = \left(\frac{z}{a_{2n+2}} + b_{2n+2}\right)\tilde{R}_{2n+1}(z) + \lambda_{2n+2}\tilde{R}_{2n}(z),$$

for all $n \geqslant 0$. *Then,*

(1) $\{\tilde{R}_n\}_{n\geqslant 0}$ *is a monic regular sequence of L-polynomials.*
(2) *There exists a SMF* $\widehat{\mathscr{L}}$ *which is unique apart from a nonzero constant factor, for which* $\{\tilde{R}_n\}_{n\geqslant 0}$ *is the corresponding monic regular OLPS if and only if* $\lambda_n \neq 0$, *for all* $n \in \mathbb{N}$.
Furthermore, the SMF $\widehat{\mathscr{L}}$ *is positive definite if and only if,*

$$\frac{\lambda_n\, a_n}{b_n} > 0, \quad n = 1, 2, \ldots .$$

We are now in a position to prove the following.

Theorem 12. *Let* $\mathscr{L}$ *be the SMF associated with Dawson's integral given by* (21) *according to the ordering induced by* $\{p(n)\}_{n\geqslant 0}$ *such that* $p(n) = E[(n+1)/2]$. *Then,* $\mathscr{L}$ *is quasi-definite and any OLPS* $\{R_n\}_{n\geqslant 0}$ *is regular and has the form* $R_n(z) = \gamma_n z^{-p(n)} B_n(z)$, $\gamma_n \neq 0$, *with* B_n *satisfying* (24). *Furthermore,* $\mathscr{L}$ *cannot be positive definite.*

Proof. Let $\{B_n\}_{n\geqslant 0}$ be the sequence of denominators of the $[n/n]$-2PA to (G_0, G_∞). As we have already seen from (24), B_n is a monic polynomial of degree n and $B_n(0) \neq 0$ for $n = 0, 1, \ldots$. In order to construct a sequence of monic regular L-polynomials, consider

$$Q_n(z) = \begin{cases} B_n(z) & \text{when } n \text{ is even,} \\ \dfrac{B_n(z)}{B_n(0)} & \text{when } n \text{ is odd.} \end{cases}$$

Hence, the sequence $\{R_n\}_{n\geqslant 0}$ given by

$$R_n(z) = \frac{Q_n(z)}{z^{p(n)}}, \quad p(n) = \mathrm{E}[(n+1)/2],$$

represents a monic regular sequence of L-polynomials for the ordering induced by $p(n) = \mathrm{E}[(n+1)/2]$.

Now, from (24) and (25), one can write

$$\frac{Q_{2n+1}(z)}{z^{n+1}} = (z + \beta_{2n+1})\frac{Q_{2n}(z)}{z^{n+1}B_{2n+1}(0)} + \frac{\alpha_{2n+1}}{\beta_{2n+1}\beta_{2n}}\frac{Q_{2n-1}(z)}{z^n},$$

yielding

$$R_{2n+1}(z) = (z + \beta_{2n+1})\, z^{-1}\frac{R_{2n}(z)}{B_{2n+1}(0)} + \lambda_{2n+1}R_{2n-1}(z),$$

with $\lambda_{2n+1} = \frac{\alpha_{2n+1}}{\beta_{2n+1}\beta_{2n}} \neq 0$ by (23), or

$$R_{2n+1}(z) = \left(\frac{1}{B_{2n+1}(0)} + \frac{z^{-1}}{B_{2n}(0)}\right)R_{2n}(z) + \lambda_{2n+1}R_{2n-1}(z). \tag{26}$$

Similarly, for the even case, again from (24),

$$\frac{Q_{2n+2}(z)}{z^{n+1}} = (z+\beta_{2n+2})\, B_{2n+1}(0)\, \frac{Q_{2n+1}(z)}{z^{n+1}} + \alpha_{2n+2}\, z\, \frac{Q_{2n}(z)}{z^{n+1}}.$$

Hence, there holds

$$R_{2n+2}(z) = (z+\beta_{2n+2})B_{2n+1}(0)R_{2n+1}(z) + \lambda_{2n+2}R_{2n}(z), \tag{27}$$

where $\lambda_{2n+2} = \alpha_{2n+2} \neq 0$ by (23). Now, (27) can be expressed as (recall $B_n(0) = \beta_1 \ldots \beta_n > 0$)

$$R_{2n+2}(z) = (zB_{2n+1}(0) + B_{2n+2}(0))R_{2n+1}(z) + \lambda_{2n+2}R_{2n}(z). \tag{28}$$

Now, set

$$a_1 = 1, \quad a_{2n+1} = B_{2n}(0), \;\; n = 1, 2, \ldots$$
$$a_{2n+2} = \frac{1}{B_{2n+1}(0)}, \;\; n = 0, 1, \ldots$$
$$b_n = a_{n+1}, \;\; n = 0, 1, \ldots .$$

Thus, $b_{2n+1} = a_{2n+2} = (B_{2n+1}(0))^{-1} \neq 0$, $b_{2n+2} = a_{2n+3} = B_{2n+2}(0) \neq 0$. With these notation, (26) and (28) can be rewritten as

$$R_{2n+1}(z) = \left(\frac{z^{-1}}{a_{2n+1}} + b_{2n+1}\right) R_{2n}(z) + \lambda_{2n+1}R_{2n-1}(z),$$
$$R_{2n+2}(z) = \left(\frac{z}{a_{2n+2}} + b_{2n+2}\right) R_{2n+1}(z) + \lambda_{2n+2}R_{2n}(z).$$

Now, making use of Theorem 11, there exists a quasi-definite SMF $\widehat{\mathscr{L}}$ for which $\{R_n\}_{n\geqslant 0}$ is the corresponding monic regular OLPS. Furthermore, taking into account that $a_n > 0$, $b_n > 0$ and that $\lambda_n < 0$, $\widehat{\mathscr{L}}$ cannot be positive definite. Thus, it only remains to prove that $\mathscr{L} = \widehat{\mathscr{L}}$. Since $\{R_n\}_{n\geqslant 0}$ is a basis for Λ, we shall prove that $\mathscr{L}(R_n) = \widehat{\mathscr{L}}(R_n)$, for $n = 0, 1, \ldots$. Now, we know that $\widehat{\mathscr{L}}(R_n) = 0$, for $n = 1, 2 \ldots$ and by Remark 10, $\mathscr{L}(R_n) = 0$, for $n = 1, 2 \ldots$. Since $R_0 \equiv 1$, from the proof of Theorem 11 (see [5, p. 73]) one has

$$\widehat{\mathscr{L}}(R_0) = \widehat{\mathscr{L}}(1) = \frac{a_1}{b_1} = \frac{a_1}{a_2} = \frac{1}{B_1(0)} = \frac{1}{\beta_1} = 1.$$

Actually, this is a matter of normalization since it suffices to take $\widehat{\mathscr{L}}(R_0) \neq 0$. On the other hand, from (21),

$$\mathscr{L}(R_0) = \mathscr{L}(1) = \mu_0 = c_1^* = 1.$$

Hence, we see that both functionals coincide and the proof follows. □

In the rest of this section, we shall be concerned with the location of the zeros of the OLPS for the functional $\mathscr{L}$ associated with the Dawson's integral, which is not positive definite. To this respect, we illustrate graphically this fact by drawing the zeros of R_n for $n = 1, 2, \ldots 10$, as shown in Fig. 1. Next,

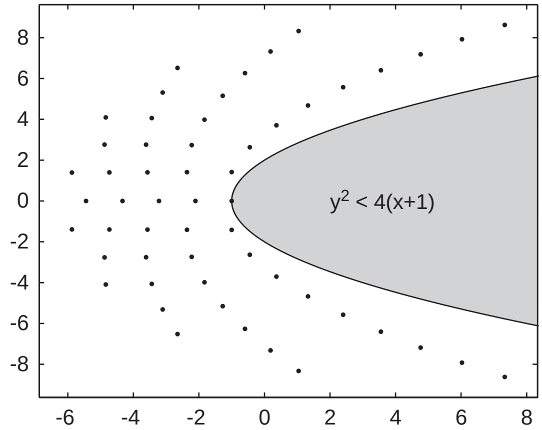

Fig. 1. Zeros of R_n, $n = 1, 2, \ldots, 10$.

a first result will be deduced as a consequence of the convergence of the sequence $\{[n/n]_{(G_0,G_\infty)}\}$ to G. Indeed, one has,

Theorem 13. *The sequence* $\{[n/n]_{(G_0,G_\infty)}\}_{n\geqslant 0}$ *converges to* G *uniformly on compacts of the finite complex plane. Furthermore, convergence is more than geometric in the sense that*

$$\lim_{n\to\infty} |G(z) - [n/n]_{(G_0,G_\infty)}(z)|^{\frac{1}{n}} = 0.$$

Proof. Let D_n be the continued fraction associated with Dawson's integral, i.e.

$$D(z) = \frac{z}{1+2z^2} - \frac{4z^2}{3+2z^2} - \frac{8z^2}{5+2z^2} - \ldots - \frac{4(n-1)z^2}{2n-1+2z^2} - \ldots$$

and let D_n denote its nth convergent, then from [16] there holds,

$$D(z) - D_n(z) = \frac{\pi\, n!\, \exp(-2z^2)\, z^{2n+1}\, {}_1F_1\left(\frac{1}{2}, n+\frac{3}{2}, z^2\right)}{2\,\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\, {}_1F_1\left(\frac{1}{2}, \frac{1}{2}-n, -z^2\right)}.$$

Since $D(z) = zG(2z^2)$ and also $D_n(z) = zG_n(2z^2)$, one has

$$zG(2z^2) - zG_n(2z^2) = \frac{z\,\pi\, n!\, \exp(-2z^2)\, z^{2n}\, {}_1F_1\left(\frac{1}{2}, n+\frac{3}{2}, \frac{2z^2}{2}\right)}{(2n+1)\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2 {}_1F_1\left(\frac{1}{2}, \frac{1}{2}-n, -\frac{2z^2}{2}\right)}.$$

Setting $2z^2 = x$, one obtains,

$$E_n(x) = G(x) - G_n(x) = \frac{\pi\, n!\, \exp(-x)\, x^n\, {}_1F_1\left(\frac{1}{2}, n+\frac{3}{2}, \frac{x}{2}\right)}{2^n(2n+1)\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2 {}_1F_1\left(\frac{1}{2}, \frac{1}{2}-n, -\frac{x}{2}\right)}.$$

Now, making use of known asymptotic estimates of the hypergeometric function, as one or more of the parameters becomes large, one finds,

$$E_n(x) \sim \gamma_n(x) = \frac{\pi\, n!\, x^n \exp(-x)}{(2n+1)2^n\left[\Gamma\left(n+\frac{1}{2}\right)\right]^2} \quad (n \to \infty). \tag{29}$$

Finally, making use of the Stirling formula, i.e.

$$\Gamma(p+1) \sim \sqrt{2\pi p}\, p^p \exp(-p) \quad (p \to \infty),$$

it can be easily checked that $\lim_{n\to\infty}|\gamma_n(x)| = 0$ uniformly on compact sets of $\mathbb{C}$ and, furthermore, that $\lim_{n\to\infty}|\gamma_n(x)|^{1/n} = 0$. Thus, from (29) proof follows. □

As a consequence and taking into account that $G(z) \neq 0$ for all $z \in \mathbb{C}$, one has the following.

Corollary 14. *Let K be an arbitrary compact subset of the finite complex plane. Then, there exists a natural number n_0 depending on K, such that for any $n > n_0$, the zeros of R_n lie outside K.*

A sharper result that enables us to more precisely justify the graphical behavior exhibited in Fig. 1, will be now established.

Theorem 15. *Let $\{R_n\}_{n\geqslant 0}$ be an OLPS for the SMF $\mathscr{L}$ associated with Dawson's integral with respect to the ordering induced by $\{p(n)\}_{n\geqslant 0}$ such that $p(n) = E[(n+1)/2]$. Then, for $n = 1, 2, \ldots$, the zeros of R_n are outside the parabolic region*

$$\mathscr{P} = \{z = x + \mathrm{i}y \in \mathbb{C} : y^2 \leqslant 4(x+1),\ x > -1\}. \tag{30}$$

Proof. Since, up to a multiplicative factor, one can write $R_n(z) = z^{-p(n)}B_n(z)$, $p(n) = \mathrm{E}[(n+1)/2]$, B_n being the denominator of $[n/n]_{(G_0,G_\infty)}$, we shall concentrate ourselves on the location of the zeros of the polynomials B_n, $n = 1, 2, \ldots$. For this purpose, set

$$\widetilde{B}_n(z) = \frac{B_n(z)}{B_n(0)}, \quad (\widetilde{B}_n(0) = 1).$$

Then, from (24) one has for $n \geqslant 0$ ($B_{n+1}(0) = \beta_{n+1}B_n(0)$),

$$\frac{B_{n+1}(z)}{B_{n+1}(0)} = (z + \beta_{n+1})\frac{B_n(z)}{B_{n+1}(0)} + \alpha_{n+1}\, z\, \frac{B_{n-1}(z)}{B_{n+1}(0)},$$

yielding,

$$\widetilde{B}_{n+1}(z) = \left(1 + \frac{z}{\beta_{n+1}}\right)\widetilde{B}_n(z) - \frac{|\alpha_{n+1}|}{\beta_n\beta_{n+1}}\, z\, \widetilde{B}_{n-1}(z), \quad n \geqslant 1$$

(recall that for $n \geqslant 2$, $\alpha_n = -2(n-1)$), or equivalently,

$$\widetilde{B}_n(z) = \left(1 + \frac{z}{\beta_n}\right)\widetilde{B}_{n-1}(z) - \frac{z}{C_n}\widetilde{B}_{n-2}(z), \quad n \geqslant 2, \tag{31}$$

where $C_n = \beta_n\beta_{n-1}|\alpha_n|^{-1}$, for $n \geqslant 2$. Now, since $\widetilde{B}_{-1}(z) \equiv 0$, (31) is also true for $n = 1$ by choosing C_1 as any arbitrary nonzero number. For our purpose, C_1 will be taken positive.

Now, from (31), the proof will be deduced as a direct consequence of a parabola theorem (see [23]). Indeed, let $\{p_k\}_{k\geqslant 0}$ be a sequence of polynomials of respective degree k which satisfy the three-term recurrence relation

$$p_k(z) = \left(\frac{z}{b_k} + 1\right) p_{k-1}(z) - \frac{z}{d_k}\, p_{k-2}(z), \quad k = 1, 2, \ldots, n,$$

where the b_k's and d_k's are positive real numbers for $1\leqslant k\leqslant n$, and where $p_{-1} \equiv 0$ and $p_0 \equiv P_0 \neq 0$. Set $\alpha = \alpha(n) = \min\{b_k(1 - b_{k-1}d_k^{-1}) : k = 1, 2, \ldots, n,\ b_0 = 0\}$. Then, if $\alpha > 0$, the parabolic region

$$\mathscr{P} = \{z = x + \mathrm{i}y \in \mathbb{C} : y^2 \leqslant 4\alpha(x + 1),\ x > -\alpha\}, \tag{32}$$

contains no zero of $p_1, p_2, \ldots, p_n$. By applying this theorem to the sequence $\{\widetilde{B}_k\}_{k=0}^{n}$, proof follows just taking into account that now, for each $n = 1, 2, \ldots,$

$$\alpha = \alpha(n) = \min\{\beta_k(1 - \beta_{k-1}C_k^{-1}) : k = 1, 2, \ldots, n\} = 1,$$

where $\beta_0 = 0$ and $C_1 > 0$. $\square$

As a consequence, there also holds,

Corollary 16. *For each $n = 1, 2, \ldots$, the poles of the $[n/n]$-2PA to the pair (G_0, G_∞) given by* (19) *and* (20) *are outside the parabolic region* (30).

As for the numerator A_n of $[n/n]_{(G_0,G_\infty)}$, a similar result concerning its zeros can be also deduced. Indeed, set $S_n(z) = A_{n+1}(z)$ for $n \geqslant -1$, so that $S_{-1}(z) = A_0(z) \equiv 0$ and $S_0(z) = A_1(z) = \alpha_1 z A_{-1}(z) = \alpha_1 z(z)^{-1} = \alpha_1 = 1$. Thus, for $n \geqslant 0$, S_n is a polynomial of exact degree n satisfying (recall (24))

$$S_{n+1}(z) = (z + \beta_{n+2})S_n(z) + \alpha_{n+2} z S_{n-1}(z), \quad n \geqslant 0. \tag{33}$$

Furthermore, $S_{n+1}(0) = \beta_{n+2}S_n(0) = \beta_{n+2}\beta_{n+1}S_{n-1}(0) > 0$. Define $\widetilde{S}_n(z) = S_n(z)/S_n(0)$. Thus, (33) becomes

$$\widetilde{S}_n(z) = \left(1 + \frac{z}{\beta_{n+1}}\right)\widetilde{S}_{n-1}(z) - \frac{z}{C_n}\widetilde{S}_{n-2}(z), \quad n \geqslant 1,$$

where $C_n = \beta_n\beta_{n+1}|\alpha_{n+1}|^{-1} > 0$, for $n \geqslant 1$. Hence, by applying the above parabola theorem, it follows,

Corollary 17. *For each $n = 1, 2, \ldots$, the zeros of the $[n/n]$-2PA to the pair (G_0, G_∞) given by* (19) *and* (20) *are outside the parabolic region* (30).

An illustration of Corollaries 16 and 17 is given in the Fig. 2.

Remark 18. When concerning the estimation of Dawson's integral D, in practice x is taken as a positive number. So, when using a rational approximant, say D_n to estimate D, it would be desirable that D_n has no poles in the positive real axis. Certainly, from Corollary 16, this can be assured when considering as approximations to D those ones given by $D_n(x) = xG_n(2x^2)$, G_n being the $[n/n]_{(G_0,G_\infty)} - 2$PA to G. Actually, from its own definition, the poles and zeros of D_n cannot be positive real numbers for each n, apart $x = 0$ that is clearly a zero.

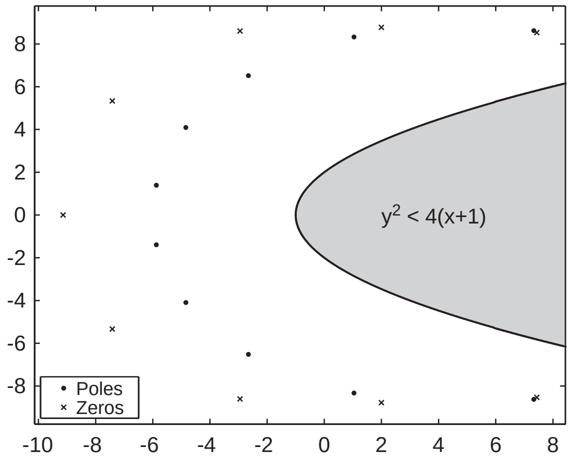

Fig. 2. Zeros and poles of $[10/10]_{(G_0,G_\infty)}$.

We conclude this section with a result where the existence of a new SMF quasi-definite and not positive definite is again deduced. Indeed, there holds

Theorem 19. *Let $\mathscr{L}$ be the SMF associated with Dawson's integral and let $\{R_n\}_{n\geqslant 0}$ be the monic OLPS. Define,*

$$\widehat{R}_n(z) = z^{s(n+1)}\mathscr{L}\left(\frac{R_{n+1}(z)-R_{n+1}(x)}{z-x}\right), \quad n=0,1,\ldots,$$

where as usual $s(n)=p(n)-p(n-1)$ and $p(n)=\mathrm{E}[(n+1)/2]$. Then,

(a) *$\{\widehat{R}_n\}_{n\geqslant 0}$ is a regular sequence of L-polynomials for the ordering induced by $\{p(n)\}_{n\geqslant 0}$.*
(b) *There exists a unique up to normalization quasi-definite SMF, $\widehat{\mathscr{L}}$, for which $\{\widehat{R}_n\}_{n\geqslant 0}$ is an OLPS.*
(c) *$\widehat{\mathscr{L}}$ cannot be positive definite.*

Proof. Set $[n/n]_{(G_0,G_\infty)}=A_n/B_n$, and $R_n(z)=z^{-p(n)}B_n(z)\in\mathscr{R}_n$. Then, by Theorem 7, one can write

$$A_n(z)=z^{p(n)}\mathscr{L}\left(\frac{R_n(z)-R_n(x)}{z-x}\right)\in\Pi_{n-1}.$$

Now, for $n\geqslant 0$ we consider again $S_n(z)=A_{n+1}(z)$, so that, by (33) along with the initial conditions $S_{-1}(z)\equiv 0$ and $S_0(z)\equiv 1$, we see that S_n is a polynomial of exact degree n and $S_n(0)\neq 0$, $n=0,1,2\ldots$, and consequently, $z^{-p(n)}S_n(z)$ represents a regular sequence of L-polynomials for the ordering induced by $\{p(n)\}_{n\geqslant 0}$.

On the other hand,

$$\frac{S_n(z)}{z^{p(n)}} = \frac{A_{n+1}(z)}{z^{p(n)}} = \frac{z^{p(n+1)}}{z^{p(n)}} \mathscr{L}\left(\frac{R_{n+1}(z) - R_{n+1}(x)}{z-x}\right)$$
$$= z^{s(n+1)} \mathscr{L}\left(\frac{R_{n+1}(z) - R_{n+1}(x)}{z-x}\right)$$
$$= \widehat{R}_n(z),$$

and part (a) follows.

As for (b) and (c), we proceed as in the proof of Theorem 12 making use of the Favard's Theorem ([5]). □

5. Some concluding remarks

As already seen in Section 2, the definition of a quasi-definite SMF depends on the fixed ordering induced by a given sequence of nonnegative integers $\{p(n)\}_{n\geqslant 0}$. Thus, when taking $p(n) = \mathrm{E}[(n+1)/2]$, we have shown that the SMF associated with Dawson's integral is quasi-definite but not positive definite. Now, one could wonder what happens for other possible orderings concerning other selections of $\{p(n)\}_{n\geqslant 0}$ and what about the location of the zeros of the corresponding sequence of orthogonal Laurent polynomials (if it exists). On the other hand, from $\{p(n)\}_{n\geqslant 0}$ we have a sequence of two-point Padé approximants to the pair (G_0, G_∞) given by (19) and (20), namely, $G_n = [p(n) + p(n-1)/n]_{(G_0, G_\infty)}$, generating rational approximations $D_n(z) = zG_n(2z^2)$ to the Dawson's integral D.

Now, what about such these approximations? Certainly, the cases $p(n) = 0$, $p(n) = n$ and $p(n) = \mathrm{E}[(n+1)/2]$, $n\geqslant 0$, have been extensively considered (see e.g. [16]). However, nothing is known about the approximation resulting from other selections. According to the analysis carried out in Section 4, a tentative way to proceed might be trying to find a continued fraction G similar to (22), so that the nth

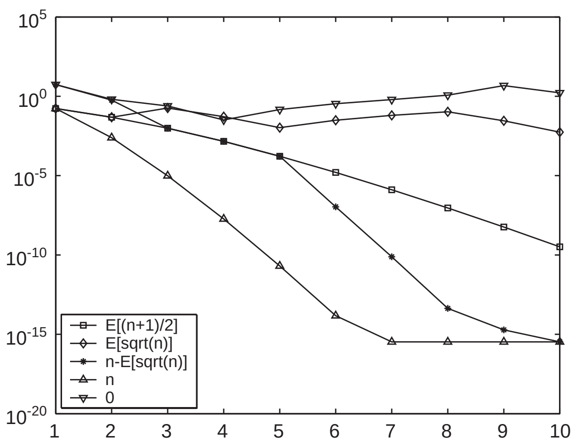

Fig. 3. Absolute error for $x = 0.75$.

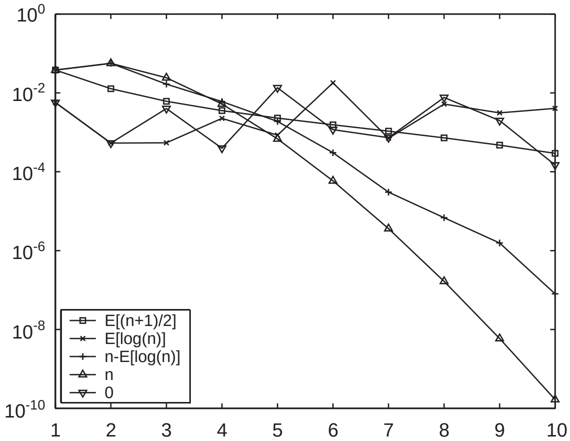

Fig. 4. Absolute error for $x = 2.5$.

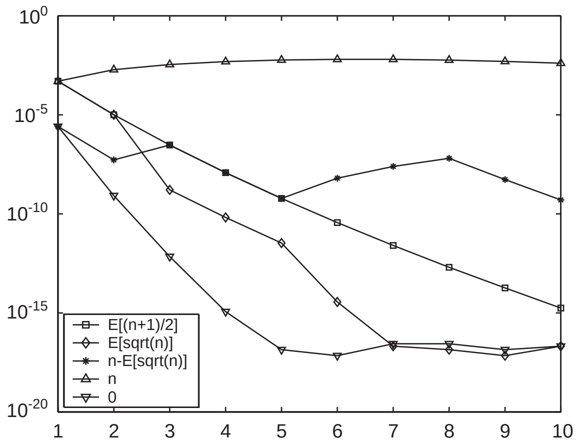

Fig. 5. Absolute error for $x = 10$.

convergent G_n coincides with the 2PA $[p(n) + p(n-1)/n]_{(G_0,G_\infty)}$. Certainly, this is an open problem to be considered in future. As an illustration, several numerical experiments have been done involving some of the following sequences of nonnegative integers: $p(n)=\mathrm{E}[(n+1)/2]$, $p(n)=\mathrm{E}[\sqrt{n}]$, $p(n)=n-\mathrm{E}[\sqrt{n}]$, $p(n) = \mathrm{E}[\log(n)]$, $p(n) = n - \mathrm{E}[\log(n)]$, $p(n) = n$ and $p(n) = 0$.

In Figs. 3, 4 and 5, the absolute errors provided by the approximations $D_n(x)$, $n = 1, 2, \ldots, n$ for different fixed values of x are shown. Thus, when trying to obtain a suitable global estimation for D, on the whole positive real half-line, it seems the most advisable candidate is that one corresponding to $p(n) = \mathrm{E}[(n + 1)/2]$. However, from the discussion above, it is also clear that a much deeper analysis needs to be carried out.

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